A Fast Linear-Arithmetic Solver for DPLL(T)

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Introduction

- Satisfiability Modulo Theories (SMT).
- SMT is the problem of determining satisfiability of formulas modulo background theories.
- Examples of background theories:
  - linear arithmetic: $x + 1 \leq y$
  - arrays: $a[i := v_1][j] = v_2$
  - uninterpreted functions: $f(f(f(x))) = x$
  - data-types: $car(cons(v_1, v_3)) = v_2$
  - bit-vectors: $concat(bv_1, bv_2) = bv_3$

Applications of SMT

- Extended Static Checking
- Equivalence Checking (Hardware)
- Bounded Model Checking
- Predicate Abstraction
- Symbolic Simulation
- Test Case Generation
- AI Planning & Scheduling
- Embedded in Theorem Provers (e.g., PVS)

SAT solvers + Decision Procedures

- This approach was independently developed by several groups: CVC (Stanford), ICS (SRI), MathSAT (Univ. Trento, Italy), Verifun (HP).
- It was motivated by the breakthroughs in SAT solving.
- SAT solver “manages” the boolean structure, and assigns truth values to the atoms in a formula.
- Decision procedure is used to validate the (partial) assignment produced by the SAT solver.
- Decision procedure detects a conflict $\longrightarrow$ a new clause (lemma) is created.

Precise Lemmas

- Lemma:

  $\{a_1 = \textsf{T}, a_1 = \textsf{F}, a_3 = \textsf{F}\}$*is inconsistent* $\rightsquigarrow \neg a_1 \vee a_2 \vee a_3$

- An inconsistent A set is redundant if $A' \subset A$ is also inconsistent.
- Redundant inconsistent sets $\rightsquigarrow$ Imprecise Lemmas $\rightsquigarrow$ Ineffective pruning of the search space.
- Noise of a redundant set: $A \setminus A_{min}$.
- The imprecise lemma is useless in any context (partial assignment) where an atom in the noise has a different assignment.
- Example: suppose a_1 is in the noise, then $\neg a_1 \vee a_2 \vee a_3$ is useless when $a_1 = F$.

Theory Propagation

- The SAT solver is assigning truth values to the atoms in a formula.
- The partial assignment produced by the SAT solver may imply the truth value of unassigned atoms.
- Example:

$$x = y \wedge y = z \wedge (f(x) \neq f(z) \vee f(x) = f(w))$$

The partial assignment $\{x = y \rightarrow \mathsf{T}, y = z \rightarrow \mathsf{T}\}$ implies $f(x) = f(z)$.

- Reduces the number of conflicts and the search space.

Efficient Backtracking

- One of the most important improvements in SAT was efficient backtracking.
- Until recently, backtracking was ignored in the design of decision procedures.
- Extreme (inefficient) approach: restart from scratch on every conflict.
- Other inefficient solutions:
  - Functional data-structures.
  - Backtrackable data-structures (trail-stack).
- Backtracking should be included in the design of the decision procedure.
- Restore to a logically equivalent state.

The ideal SMT solver

- Efficient in real benchmarks.
- Produces precise lemmas.
- Supports Theory Propagation.
- Incremental.
- Efficient Backtracking.
- Produces counterexamples.

Linear Arithmetic (LA)

- Most important theory.
- Present in most applications.
- Algorithms:
  - Graph based (e.g., Bellman-Ford, Floyd-Warshall, etc) for difference logic (DF).
  - Fourier-Motzkin elimination
  - Simplex
- Difference logic is very specialized. The interesting case is linear arithmetic.
- Challenge: efficient on LA and competitive on DF.

Standard Simplex

- Standard Form: $Ax = b$ and $x \geq 0$.
- Much more efficient than Fourier-Motzkin elimination.
- It is not competitive in DF.
- Incremental: add/remove equations (i.e., rows).
- Slow backtracking
- No theory propagation.
- Used in several solvers: Simplify, MathSAT, ICS, Simplics, Old Yices 0.1.
- Off-the-shelf simplex solvers: unsound & incomplete (floating point numbers).

Fast Linear Arithmetic

- Simplex General Form.
- New algorithm based on the Dual Simplex.
- Precise lemmas.
- Efficient Backtracking.
- Efficient Theory Propagation.
- New approach for solving strict inequalities ($t > 0$).
- Presimplification step.
- Integer problems: Gomory cuts, Branch & Bound, GCD test.
- This algorithm is used in the new Yices.
- Outperforms specialized solvers on difference logic.

General Form

- General Form: $Ax = 0$ and $l_j \leq x_j \leq u_j$
- Example:

$$x \geq 0, (x + y \leq 2 \vee x + 2y \geq 6), (x + y = 2 \vee x + 2y > 4)$$

$$\rightsquigarrow$$

$$s_1 = x + y, s_2 = x + 2y,$$

$$x \geq 0, (s_1 \leq 2 \vee s_2 \geq 6), (s_1 = 2 \vee s_2 > 4)$$

- Only bounds (e.g., $s_1 \leq 2$) are asserted during the search.
- Presimplification: Unconstrained variables can be eliminated before the beginning of the search.

Equations + Bounds + Assignment

- An assignment is a mapping from variables to values.
- We maintain an assignment that satisfies all equations and bounds.
- The assignment of non dependent variables implies the assignment of dependent variables.
- Equations + Bounds can be used to derive new bounds.
- Example: $x = y - z,\ y \leq 2,\ z \geq 3 \rightsquigarrow x \leq -1$.
  - Explanation: $y \leq 2,\ z \geq 3$
- The new bound may be inconsistent with the already known bounds.
- Example: $x \leq -1,\ x \geq 0$.

Strict Inequalities

- The method described only handles non-strict inequalities (e.g., $x \leq 2$).
- For integer problems, strict inequalities can be converted into non-strict inequalities. $x < 1 \rightsquigarrow x \leq 0$.
- For rational/real problems, strict inequalities can be converted into non-strict inequalities using a small δ. $x < 1 \rightsquigarrow x \leq 1 - \delta$.
- We do not compute a δ, we treat it symbolically.
- δ is an infinitesimal parameter: $(c, k) = c + k\delta$

Example

- Initial state

$$s \geq 1, x \geq 0$$

$$(y \leq 1 \vee v \geq 2), (v \leq -2 \vee v \geq 0), (v \leq -2 \vee u \leq -1)$$

Assignment	Equations	Bounds
$x = 0$	$s = x + y$	
$y = 0$	$u = x + 2y$	
$s = 0$	$v = x - y$	
$u = 0$		
$v = 0$		

Example

- Asserting $s \geq 1$

$$s \geq 1, x \geq 0$$

$$(y \leq 1 \vee v \geq 2), (v \leq -2 \vee v \geq 0), (v \leq -2 \vee u \leq -1)$$

Assignment	Equations	Bounds
$x = 0$	$s = x + y$	
$y = 0$	$u = x + 2y$	
$s = 0$	$v = x - y$	
$u = 0$		
$v = 0$		

Example

- Asserting $s \geq 1$ assignment does not satisfy new bound.

$$s \geq 1, x \geq 0$$

$$(y \leq 1 \vee v \geq 2), (v \leq -2 \vee v \geq 0), (v \leq -2 \vee u \leq -1)$$

Assignment	Equations	Bounds
$x = 0$	$s = x + y$	$s \geq 1$
$y = 0$	$u = x + 2y$	
$s = 0$	$v = x - y$	
$u = 0$		
$v = 0$		

Example

- Asserting $s \geq 1$ pivot s and x (s is a dependent variable).

$$s \geq 1, x \geq 0$$

$$(y \leq 1 \vee v \geq 2), (v \leq -2 \vee v \geq 0), (v \leq -2 \vee u \leq -1)$$

Assignment	Equations	Bounds
$x = 0$	$s = x + y$	$s \geq 1$
$y = 0$	$u = x + 2y$	
$s = 0$	$v = x - y$	
$u = 0$		
$v = 0$		

Example

- Asserting $s \geq 1$ pivot s and x (s is a dependent variable).

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Assignment	Equations	Bounds
$x = 0$	$x = s - y$	$s \geq 1$
$y = 0$	$u = x + 2y$	
$s = 0$	$v = x - y$	
$u = 0$		
$v = 0$		

Example

- Asserting $s \geq 1$ pivot s and x (s is a dependent variable).

$$s \geq 1, x \geq 0$$

$$(y \leq 1 \vee v \geq 2), (v \leq -2 \vee v \geq 0), (v \leq -2 \vee u \leq -1)$$

Assignment	Equations	Bounds
$x = 0$	$x = s - y$	$s \geq 1$
$y = 0$	$u = s + y$	
$s = 0$	$v = s - 2y$	
$u = 0$		
$v = 0$		

Example

- Asserting $s \geq 1$ update assignment.

$$s \geq 1, x \geq 0$$

$$(y \leq 1 \vee v \geq 2), (v \leq -2 \vee v \geq 0), (v \leq -2 \vee u \leq -1)$$

Assignment	Equations	Bounds
$x = 0$	$x = s - y$	$s \geq 1$
$y = 0$	$u = s + y$	
$s = 1$	$v = s - 2y$	
$u = 0$		
$v = 0$		

Example

- Asserting $s \geq 1$ update dependent variables assignment.

$$s \geq 1, x \geq 0$$

$$(y \leq 1 \vee v \geq 2), (v \leq -2 \vee v \geq 0), (v \leq -2 \vee u \leq -1)$$

Assignment	Equations	Bounds
$x = 1$	$x = s - y$	$s \geq 1$
$y = 0$	$u = s + y$	
$s = 1$	$v = s - 2y$	
$u = 1$		
$v = 1$		

Example

- Asserting $x \geq 0$

$$s \geq 1, x \geq 0$$

$$(y \leq 1 \vee v \geq 2), (v \leq -2 \vee v \geq 0), (v \leq -2 \vee u \leq -1)$$

Assignment	Equations	Bounds
$x = 1$	$x = s - y$	$s \geq 1$
$y = 0$	$u = s + y$	
$s = 1$	$v = s - 2y$	
$u = 1$		
$v = 1$		

Example

- Asserting $x \geq 0$ assignment satisfies new bound.

$$s \geq 1, x \geq 0$$

$$(y \leq 1 \vee v \geq 2), (v \leq -2 \vee v \geq 0), (v \leq -2 \vee u \leq -1)$$

Assignment	Equations	Bounds
$x = 1$	$x = s - y$	$s \geq 1$
$y = 0$	$u = s + y$	$x \geq 0$
$s = 1$	$v = s - 2y$	
$u = 1$		
$v = 1$		

Example

- Case split $\neg y \leq 1$

$$s \geq 1, x \geq 0$$

$$(y \leq 1 \vee v \geq 2), (v \leq -2 \vee v \geq 0), (v \leq -2 \vee u \leq -1)$$

Assignment	Equations	Bounds
$x = 1$	$x = s - y$	$s \geq 1$
$y = 0$	$u = s + y$	$x \geq 0$
$s = 1$	$v = s - 2y$	
$u = 1$		
$v = 1$		

Example

- Case split $\neg y \leq 1$ assignment does not satisfies new bound.

$$s \geq 1, x \geq 0$$

$$(y \leq 1 \vee v \geq 2), (v \leq -2 \vee v \geq 0), (v \leq -2 \vee u \leq -1)$$

Assignment	Equations	Bounds
$x = 1$	$x = s - y$	$s \geq 1$
$y = 0$	$u = s + y$	$x \geq 0$
$s = 1$	$v = s - 2y$	$y > 1$
$u = 1$		
$v = 1$		

Example

- Case split $\neg y \leq 1$ update assignment.

$$s \geq 1, x \geq 0$$

$$(y \leq 1 \vee v \geq 2), (v \leq -2 \vee v \geq 0), (v \leq -2 \vee u \leq -1)$$

Assignment	Equations	Bounds
$x = 1$	$x = s - y$	$s \geq 1$
$y = 1 + \delta$	$u = s + y$	$x \geq 0$
$s = 1$	$v = s - 2y$	$y > 1$
$u = 1$		
$v = 1$		

Example

- Case split $\neg y \leq 1$ update dependent variables assignment.

$$s \geq 1, x \geq 0$$

$$(y \leq 1 \vee v \geq 2), (v \leq -2 \vee v \geq 0), (v \leq -2 \vee u \leq -1)$$

Assignment	Equations	Bounds
$x = -\delta$	$x = s - y$	$s \geq 1$
$y = 1 + \delta$	$u = s + y$	$x \geq 0$
$s = 1$	$v = s - 2y$	$y > 1$
$u = 2 + \delta$		
$v = -1 - 2\delta$		

Example

- Bound violation

$$s \geq 1, x \geq 0$$

$$(y \leq 1 \vee v \geq 2), (v \leq -2 \vee v \geq 0), (v \leq -2 \vee u \leq -1)$$

Assignment	Equations	Bounds
$x = -\delta$	$x = s - y$	$s \geq 1$
$y = 1 + \delta$	$u = s + y$	$x \geq 0$
$s = 1$	$v = s - 2y$	$y > 1$
$u = 2 + \delta$		
$v = -1 - 2\delta$		

Example

- Bound violation pivot x and s (x is a dependent variables).

$$s \geq 1, x \geq 0$$

$$(y \leq 1 \vee v \geq 2), (v \leq -2 \vee v \geq 0), (v \leq -2 \vee u \leq -1)$$

Assignment

$$\begin{aligned} x &= -\delta \\ y &= 1+\delta \\ s &= 1 \\ u &= 2+\delta \\ v &= -1-2\delta \end{aligned}$$

Equations

$$\begin{aligned} x &= s-y \\ u &= s+y \\ v &= s-2y \end{aligned}$$

Bounds

$$\begin{array}{ccc} s & \geq & 1 \\ x & \geq & 0 \\ \hline y & > & 1 \end{array}$$

Example

- Bound violation pivot x and s (x is a dependent variables).

$$s \geq 1, x \geq 0$$

$$(y \leq 1 \vee v \geq 2), (v \leq -2 \vee v \geq 0), (v \leq -2 \vee u \leq -1)$$

Assignment	Equations	Bounds
$x = -\delta$	$s = x + y$	$s \geq 1$
$y = 1 + \delta$	$u = s + y$	$x \geq 0$
$s = 1$	$v = s - 2y$	$y > 1$
$u = 2 + \delta$		
$v = -1 - 2\delta$		

Example

- Bound violation pivot x and s (x is a dependent variables).

$$s \geq 1, x \geq 0$$

$$(y \leq 1 \vee v \geq 2), (v \leq -2 \vee v \geq 0), (v \leq -2 \vee u \leq -1)$$

Assignment	Equations	Bounds
$x = -\delta$	$s = x + y$	$s \geq 1$
$y = 1 + \delta$	$u = x + 2y$	$x \geq 0$
$s = 1$	$v = x - y$	$y > 1$
$u = 2 + \delta$		
$v = -1 - 2\delta$		

Example

- Bound violation update assignment.

$$s \geq 1, x \geq 0$$

$$(y \leq 1 \vee v \geq 2), (v \leq -2 \vee v \geq 0), (v \leq -2 \vee u \leq -1)$$

Assignment	Equations	Bounds
$x = 0$	$s = x + y$	$s \geq 1$
$y = 1 + \delta$	$u = x + 2y$	$x \geq 0$
$s = 1$	$v = x - y$	$y > 1$
$u = 2 + \delta$		
$v = -1 - 2\delta$		

Example

- Bound violation update dependent variables assignment.

$$s \geq 1, x \geq 0$$

$$(y \leq 1 \vee v \geq 2), (v \leq -2 \vee v \geq 0), (v \leq -2 \vee u \leq -1)$$

Assignment	Equations	Bounds
$x = 0$	$s = x + y$	$s \geq 1$
$y = 1 + \delta$	$u = x + 2y$	$x \geq 0$
$s = 1 + \delta$	$v = x - y$	$y > 1$
$u = 2 + 2\delta$		
$v = -1 - \delta$		

Example

- Theory propagation $x \geq 0, y > 1 \rightsquigarrow u > 2$

$$s \geq 1, x \geq 0$$

$$(y \leq 1 \vee v \geq 2), (v \leq -2 \vee v \geq 0), (v \leq -2 \vee u \leq -1)$$

Assignment	Equations	Bounds
$x = 0$	$s = x + y$	$s \geq 1$
$y = 1 + \delta$	$u = x + 2y$	$x \geq 0$
$s = 1 + \delta$	$v = x - y$	$y > 1$
$u = 2 + 2\delta$		
$v = -1 - \delta$		

Example

- Theory propagation $u > 2 \leadsto \neg u \leq -1$

$$s \geq 1, x \geq 0$$

$$(y \leq 1 \vee v \geq 2), (v \leq -2 \vee v \geq 0), (v \leq -2 \vee u \leq -1)$$

Assignment	Equations	Bounds
$x = 0$	$s = x + y$	$s \geq 1$
$y = 1 + \delta$	$u = x + 2y$	$x \geq 0$
$s = 1 + \delta$	$v = x - y$	$y > 1$
$u = 2 + 2\delta$		$u > 2$
$v = -1 - \delta$		

Example

- Boolean propagation $\neg y \leq 1 \rightsquigarrow v \geq 2$

$$s \geq 1, x \geq 0$$

$$(y \leq 1 \vee v \geq 2), (v \leq -2 \vee v \geq 0), (v \leq -2 \vee u \leq -1)$$

Assignment	Equations	Bounds
$x = 0$	$s = x + y$	$s \geq 1$
$y = 1 + \delta$	$u = x + 2y$	$x \geq 0$
$s = 1 + \delta$	$v = x - y$	$y > 1$
$u = 2 + 2\delta$		$u > 2$
$v = -1 - \delta$		

Example

- Theory propagation $v \geq 2 \rightsquigarrow \neg v \leq -2$

$$s \geq 1, x \geq 0$$

$$(y \leq 1 \vee v \geq 2), (v \leq -2 \vee v \geq 0), (v \leq -2 \vee u \leq -1)$$

Assignment

$$\begin{aligned} x &= 0 \\ y &= 1 + \delta \\ s &= 1 + \delta \\ u &= 2 + 2\delta \\ v &= -1 - \delta \end{aligned}$$

Equations

$$\begin{aligned} s &= x + y \\ u &= x + 2y \\ v &= x - y \end{aligned}$$

Bounds

$$\begin{array}{ccc} s & \geq & 1 \\ x & \geq & 0 \\ \hline y & > & 1 \\ u & > & 2 \end{array}$$

Example

- Conflict empty clause

$$s \geq 1, x \geq 0$$

$$(y \leq 1 \vee v \geq 2), (v \leq -2 \vee v \geq 0), (v \leq -2 \vee u \leq -1)$$

Assignment	Equations	Bounds
$x = 0$	$s = x + y$	$s \geq 1$
$y = 1 + \delta$	$u = x + 2y$	$x \geq 0$
$s = 1 + \delta$	$v = x - y$	$y > 1$
$u = 2 + 2\delta$		$u > 2$
$v = -1 - \delta$		

Example

- Backtracking

$$s \geq 1, x \geq 0$$

$$(y \leq 1 \vee v \geq 2), (v \leq -2 \vee v \geq 0), (v \leq -2 \vee u \leq -1)$$

Assignment	Equations	Bounds
$x = 0$	$s = x + y$	$s \geq 1$
$y = 1 + \delta$	$u = x + 2y$	$x \geq 0$
$s = 1 + \delta$	$v = x - y$	
$u = 2 + 2\delta$		
$v = -1 - \delta$		

Example

- Asserting $y \leq 1$

$$s \geq 1, x \geq 0$$

$$(y \leq 1 \vee v \geq 2), (v \leq -2 \vee v \geq 0), (v \leq -2 \vee u \leq -1)$$

Assignment	Equations	Bounds
$x = 0$	$s = x + y$	$s \geq 1$
$y = 1 + \delta$	$u = x + 2y$	$x \geq 0$
$s = 1 + \delta$	$v = x - y$	
$u = 2 + 2\delta$		
$v = -1 - \delta$		

Example

- Asserting $y \leq 1$ assignment does not satisfy new bound.

$$s \geq 1, x \geq 0$$

$$(y \leq 1 \vee v \geq 2), (v \leq -2 \vee v \geq 0), (v \leq -2 \vee u \leq -1)$$

Assignment	Equations	Bounds
$x = 0$	$s = x + y$	$s \geq 1$
$y = 1 + \delta$	$u = x + 2y$	$x \geq 0$
$s = 1 + \delta$	$v = x - y$	$y \leq 1$
$u = 2 + 2\delta$		
$v = -1 - \delta$		

Example

- Asserting $y \leq 1$ update assignment.

$$s \geq 1, x \geq 0$$

$$(y \leq 1 \vee v \geq 2), (v \leq -2 \vee v \geq 0), (v \leq -2 \vee u \leq -1)$$

Assignment

$$\begin{aligned} x &= 0 \\ y &= 1 \\ s &= 1 + \delta \\ u &= 2 + 2\delta \\ v &= -1 - \delta \end{aligned}$$

Equations

$$\begin{aligned} s &= x + y \\ u &= x + 2y \\ v &= x - y \end{aligned}$$

Bounds

$$\begin{array}{ccc} s & \geq & 1 \\ x & \geq & 0 \\ \hline y & \leq & 1 \end{array}$$

Example

- Asserting $y \leq 1$ update dependent variables assignment.

$$s \geq 1, x \geq 0$$

$$(y \leq 1 \vee v \geq 2), (v \leq -2 \vee v \geq 0), (v \leq -2 \vee u \leq -1)$$

Assignment	Equations	Bounds
$x = 0$	$s = x + y$	$s \geq 1$
$y = 1$	$u = x + 2y$	$x \geq 0$
$s = 1$	$v = x - y$	$y \leq 1$
$u = 2$		
$v = -1$		

Example

- Theory propagation $s \geq 1, y \leq 1 \rightsquigarrow v \geq -1$

$$s \geq 1, x \geq 0$$

$$(y \leq 1 \vee v \geq 2), (v \leq -2 \vee v \geq 0), (v \leq -2 \vee u \leq -1)$$

Assignment	Equations	Bounds
$x = 0$	$x = s - y$	$s \geq 1$
$y = 1$	$u = s + y$	$x \geq 0$
$s = 1$	$v = s - 2y$	$y \leq 1$
$u = 2$		
$v = -1$		

Example

- Theory propagation $v \geq -1 \rightsquigarrow \neg v \leq -2$

$$s \geq 1, x \geq 0$$

$$(y \leq 1 \vee v \geq 2), (v \leq -2 \vee v \geq 0), (v \leq -2 \vee u \leq -1)$$

Assignment	Equations	Bounds
$x = 0$	$x = s - y$	$s \geq 1$
$y = 1$	$u = s + y$	$x \geq 0$
$s = 1$	$v = s - 2y$	$y \leq 1$
$u = 2$		$v \geq -1$
$v = -1$		

Example

- Boolean propagation $\neg v \leq -2 \rightsquigarrow v \geq 0$

$$s \geq 1, x \geq 0$$

$$(y \leq 1 \vee v \geq 2), (v \leq -2 \vee v \geq 0), (v \leq -2 \vee u \leq -1)$$

Assignment

$$\begin{aligned} x &= 0 \\ y &= 1 \\ s &= 1 \\ u &= 2 \\ v &= -1 \end{aligned}$$

Equations

$$\begin{aligned} x &= s - y \\ u &= s + y \\ v &= s - 2y \end{aligned}$$

Bounds

$$\begin{array}{ccc} s & \geq & 1 \\ x & \geq & 0 \\ \hline y & \leq & 1 \\ v & \geq & -1 \end{array}$$

Example

- Bound violation assignment does not satisfy new bound.

$$s \geq 1, x \geq 0$$

$$(y \leq 1 \vee v \geq 2), (v \leq -2 \vee v \geq 0), (v \leq -2 \vee u \leq -1)$$

Assignment	Equations	Bounds
$x = 0$	$x = s - y$	$s \geq 1$
$y = 1$	$u = s + y$	$x \geq 0$
$s = 1$	$v = s - 2y$	$y \leq 1$
$u = 2$		$v \geq 0$
$v = -1$		

Example

- Bound violation pivot u and s (u is a dependent variable).

$$s \geq 1, x \geq 0$$

$$(y \leq 1 \vee v \geq 2), (v \leq -2 \vee v \geq 0), (v \leq -2 \vee u \leq -1)$$

Assignment	Equations	Bounds
$x = 0$	$x = s - y$	$s \geq 1$
$y = 1$	$u = s + y$	$x \geq 0$
$s = 1$	$v = s - 2y$	$y \leq 1$
$u = 2$		$v \geq 0$
$v = -1$		

Example

- Bound violation pivot u and s (u is a dependent variable).

$$s \geq 1, x \geq 0$$

$$(y \leq 1 \vee v \geq 2), (v \leq -2 \vee v \geq 0), (v \leq -2 \vee u \leq -1)$$

Assignment	Equations	Bounds
$x = 0$	$x = s - y$	$s \geq 1$
$y = 1$	$u = s + y$	$x \geq 0$
$s = 1$	$s = v + 2y$	$y \leq 1$
$u = 2$		$v \geq 0$
$v = -1$		

Example

- Bound violation pivot u and s (u is a dependent variable).

$$s \geq 1, x \geq 0$$

$$(y \leq 1 \vee v \geq 2), (v \leq -2 \vee v \geq 0), (v \leq -2 \vee u \leq -1)$$

Assignment	Equations	Bounds
$x = 0$	$x = v + y$	$s \geq 1$
$y = 1$	$u = v + 3y$	$x \geq 0$
$s = 1$	$s = v + 2y$	$y \leq 1$
$u = 2$		$v \geq 0$
$v = -1$		

Example

- Bound violation update assignment.

$$s \geq 1, x \geq 0$$

$$(y \leq 1 \vee v \geq 2), (v \leq -2 \vee v \geq 0), (v \leq -2 \vee u \leq -1)$$

Assignment	Equations	Bounds
$x = 0$	$x = v + y$	$s \geq 1$
$y = 1$	$u = v + 3y$	$x \geq 0$
$s = 1$	$s = v + 2y$	$y \leq 1$
$u = 2$		$v \geq 0$
$v = 0$		

Example

- Bound violation update dependent variables assignment.

$$s \geq 1, x \geq 0$$

$$(y \leq 1 \vee v \geq 2), (v \leq -2 \vee v \geq 0), (v \leq -2 \vee u \leq -1)$$

Assignment	Equations	Bounds
$x = 1$	$x = v + y$	$s \geq 1$
$y = 1$	$u = v + 3y$	$x \geq 0$
$s = 2$	$s = v + 2y$	$y \leq 1$
$u = 3$		$v \geq 0$
$v = 0$		

Example

- Boolean propagation $\neg v \leq -2 \rightsquigarrow u \leq -1$

$$s \geq 1, x \geq 0$$

$$(y \leq 1 \vee v \geq 2), (v \leq -2 \vee v \geq 0), (v \leq -2 \vee u \leq -1)$$

Assignment	Equations	Bounds
$x = 1$	$x = v + y$	$s \geq 1$
$y = 1$	$u = v + 3y$	$x \geq 0$
$s = 2$	$s = v + 2y$	$y \leq 1$
$u = 3$		$v \geq 0$
$v = 0$		

Example

- Bound violation   assignment does not satisfy new bound.

$$s \geq 1, x \geq 0$$

$$(y \leq 1 \vee v \geq 2), (v \leq -2 \vee v \geq 0), (v \leq -2 \vee u \leq -1)$$

Assignment	Equations	Bounds
$x = 1$	$x = v + y$	$s \geq 1$
$y = 1$	$u = v + 3y$	$x \geq 0$
$s = 2$	$s = v + 2y$	$y \leq 1$
$u = 3$		$v \geq 0$
$v = 0$		$u \leq -1$

Example

- Bound violation pivot u and y (u is a dependent variable).

$$s \geq 1, x \geq 0$$

$$(y \leq 1 \vee v \geq 2), (v \leq -2 \vee v \geq 0), (v \leq -2 \vee u \leq -1)$$

Assignment	Equations	Bounds
$x = 1$	$x = v + y$	$s \geq 1$
$y = 1$	$u = v + 3y$	$x \geq 0$
$s = 2$	$s = v + 2y$	$y \leq 1$
$u = 3$		$v \geq 0$
$v = 0$		$u \leq -1$

Example

- Bound violation pivot u and y (u is a dependent variable).

$$s \geq 1, x \geq 0$$

$$(y \leq 1 \vee v \geq 2), (v \leq -2 \vee v \geq 0), (v \leq -2 \vee u \leq -1)$$

Assignment	Equations	Bounds
$x = 1$	$x = v + y$	$s \geq 1$
$y = 1$	$y = \frac{1}{3}u - \frac{1}{3}v$	$x \geq 0$
$s = 2$	$s = v + 2y$	$y \leq 1$
$u = 3$		$v \geq 0$
$v = 0$		$u \leq -1$

Example

- Bound violation pivot u and y (u is a dependent variable).

$$s \geq 1, x \geq 0$$

$$(y \leq 1 \vee v \geq 2), (v \leq -2 \vee v \geq 0), (v \leq -2 \vee u \leq -1)$$

Assignment	Equations	Bounds
$x = 1$	$x = \frac{1}{3}u + \frac{2}{3}v$	$s \geq 1$
$y = 1$	$y = \frac{1}{3}u - \frac{1}{3}v$	$x \geq 0$
$s = 2$	$s = \frac{2}{3}u + \frac{1}{3}v$	$y \leq 1$
$u = 3$		$v \geq 0$
$v = 0$		$u \leq -1$

Example

- Bound violation update assignment.

$$s \geq 1, x \geq 0$$

$$(y \leq 1 \vee v \geq 2), (v \leq -2 \vee v \geq 0), (v \leq -2 \vee u \leq -1)$$

Assignment	Equations	Bounds
$x = 1$	$x = \frac{1}{3}u + \frac{2}{3}v$	$s \geq 1$
$y = 1$	$y = \frac{1}{3}u - \frac{1}{3}v$	$x \geq 0$
$s = 2$	$s = \frac{2}{3}u + \frac{1}{3}v$	$y \leq 1$
$u = -1$		$v \geq 0$
$v = 0$		$u \leq -1$

Example

- Bound violation update dependent variables assignment.

$$s \geq 1, x \geq 0$$

$$(y \leq 1 \vee v \geq 2), (v \leq -2 \vee v \geq 0), (v \leq -2 \vee u \leq -1)$$

Assignment	Equations	Bounds
$x = -\frac{1}{3}$	$x = \frac{1}{3}u + \frac{2}{3}v$	$s \geq 1$
$y = -\frac{1}{3}$	$y = \frac{1}{3}u - \frac{1}{3}v$	$x \geq 0$
$s = -\frac{2}{3}$	$s = \frac{2}{3}u + \frac{1}{3}v$	$y \leq 1$
$u = -1$		$v \geq 0$
$v = 0$		$u \leq -1$

Example

- Bound violations

$$s \geq 1, x \geq 0$$

$$(y \leq 1 \vee v \geq 2), (v \leq -2 \vee v \geq 0), (v \leq -2 \vee u \leq -1)$$

Assignment	Equations	Bounds
$x = -\frac{1}{3}$	$x = \frac{1}{3}u + \frac{2}{3}v$	$s \geq 1$
$y = -\frac{1}{3}$	$y = \frac{1}{3}u - \frac{1}{3}v$	$x \geq 0$
$s = -\frac{2}{3}$	$s = \frac{2}{3}u + \frac{1}{3}v$	$y \leq 1$
$u = -1$		$v \geq 0$
$v = 0$		$u \leq -1$

Example

- Bound violations pivot s and v (s is a dependent variable).

$$s \geq 1, x \geq 0$$

$$(y \leq 1 \vee v \geq 2), (v \leq -2 \vee v \geq 0), (v \leq -2 \vee u \leq -1)$$

Assignment	Equations	Bounds
$x = -\frac{1}{3}$	$x = \frac{1}{3}u + \frac{2}{3}v$	$s \geq 1$
$y = -\frac{1}{3}$	$y = \frac{1}{3}u - \frac{1}{3}v$	$x \geq 0$
$s = -\frac{2}{3}$	$s = \frac{2}{3}u + \frac{1}{3}v$	$y \leq 1$
$u = -1$		$v \geq 0$
$v = 0$		$u \leq -1$

Example

- Bound violations pivot s and v (s is a dependent variable).

$$s \geq 1, x \geq 0$$

$$(y \leq 1 \vee v \geq 2), (v \leq -2 \vee v \geq 0), (v \leq -2 \vee u \leq -1)$$

Assignment	Equations	Bounds
$x = -\frac{1}{3}$	$x = \frac{1}{3}u + \frac{2}{3}v$	$s \geq 1$
$y = -\frac{1}{3}$	$y = \frac{1}{3}u - \frac{1}{3}v$	$x \geq 0$
$s = -\frac{2}{3}$	$v = 3s - 2u$	$y \leq 1$
$u = -1$		$v \geq 0$
$v = 0$		$u \leq -1$

Example

- Bound violations pivot s and v (s is a dependent variable).

$$s \geq 1, x \geq 0$$

$$(y \leq 1 \vee v \geq 2), (v \leq -2 \vee v \geq 0), (v \leq -2 \vee u \leq -1)$$

Assignment	Equations	Bounds
$x = -\frac{1}{3}$	$x = 2s - u$	$s \geq 1$
$y = -\frac{1}{3}$	$y = -s + u$	$x \geq 0$
$s = -\frac{2}{3}$	$v = 3s - 2u$	$y \leq 1$
$u = -1$		$v \geq 0$
$v = 0$		$u \leq -1$

Example

- Bound violations update assignment.

$$s \geq 1, x \geq 0$$

$$(y \leq 1 \vee v \geq 2), (v \leq -2 \vee v \geq 0), (v \leq -2 \vee u \leq -1)$$

Assignment	Equations	Bounds
$x = -\frac{1}{3}$	$x = 2s - u$	$s \geq 1$
$y = -\frac{1}{3}$	$y = -s + u$	$x \geq 0$
$s = 1$	$v = 3s - 2u$	$y \leq 1$
$u = -1$		$v \geq 0$
$v = 0$		$u \leq -1$

Example

- Bound violations update dependent variables assignment.

$$s \geq 1, x \geq 0$$

$$(y \leq 1 \vee v \geq 2), (v \leq -2 \vee v \geq 0), (v \leq -2 \vee u \leq -1)$$

Assignment	Equations	Bounds
$x = 3$	$x = 2s - u$	$s \geq 1$
$y = -2$	$y = -s + u$	$x \geq 0$
$s = 1$	$v = 3s - 2u$	$y \leq 1$
$u = -1$		$v \geq 0$
$v = 5$		$u \leq -1$

Example

- Found satisfying assignment

$$s \geq 1, x \geq 0$$

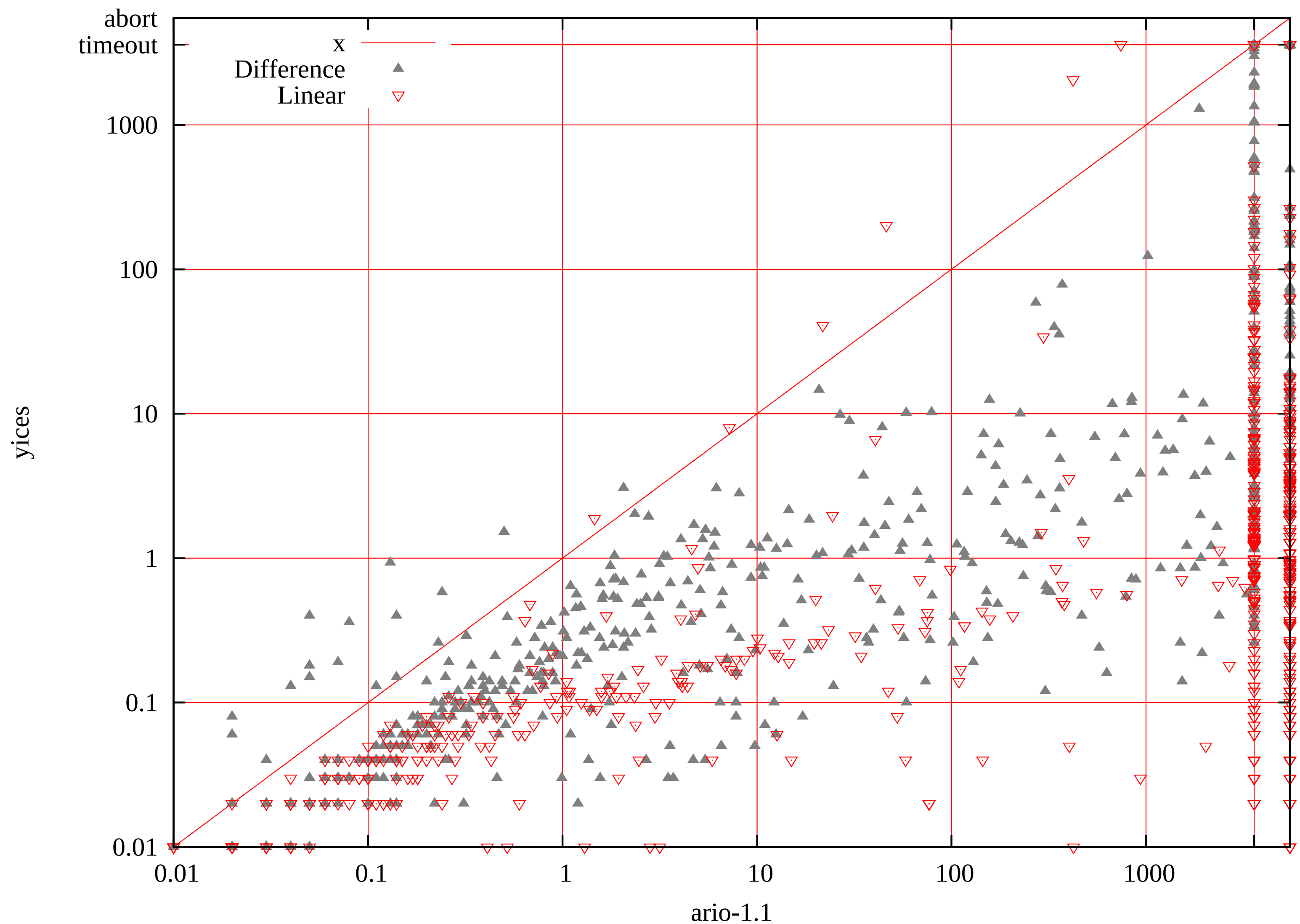
$$(y \leq 1 \vee v \geq 2), (v \leq -2 \vee v \geq 0), (v \leq -2 \vee u \leq -1)$$

Assignment	Equations	Bounds
$x = 3$	$x = 2s - u$	$s \geq 1$
$y = -2$	$y = -s + u$	$x \geq 0$
$s = 1$	$v = 3s - 2u$	$y \leq 1$
$u = -1$		$v \geq 0$
$v = 5$		$u \leq -1$

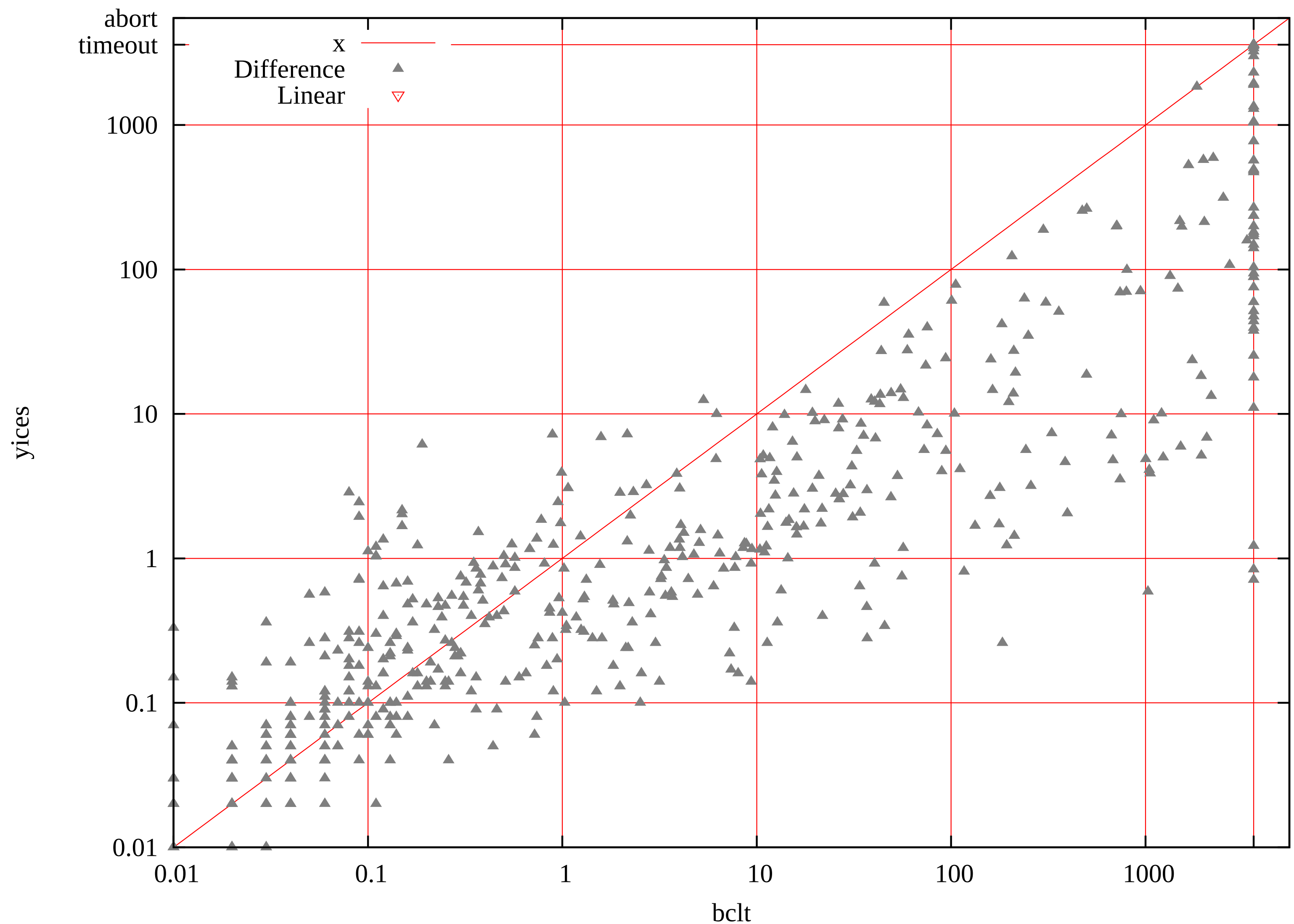
Experimental Results

- The new algorithm is used in Yices 1.0.
- We compared our new solver with:
  - Ario 1.1
  - BarcelogicTools 1.0
  - CVC Lite 2.0
  - MathSAT 3.3.1
  - Old Yices (submitted to SMT-COMP'05)
- We used all SMT-LIB benchmarks available at the time.
  - `http://goedel.cs.uiowa.edu/smtlib/`
- Timeout: 1 hour, Max. Memory: 1Gb.

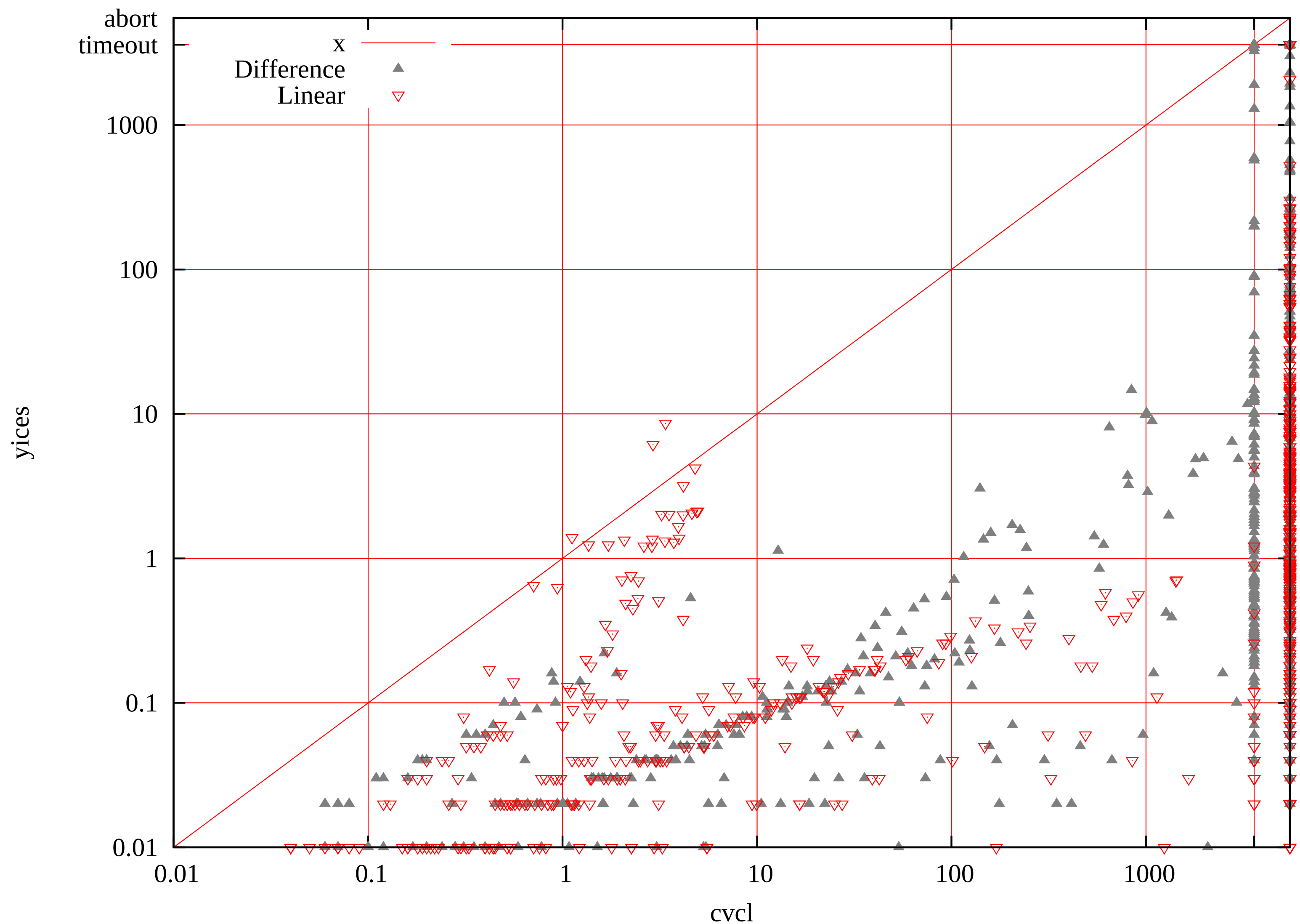
Ario 1.1 vs. Yices



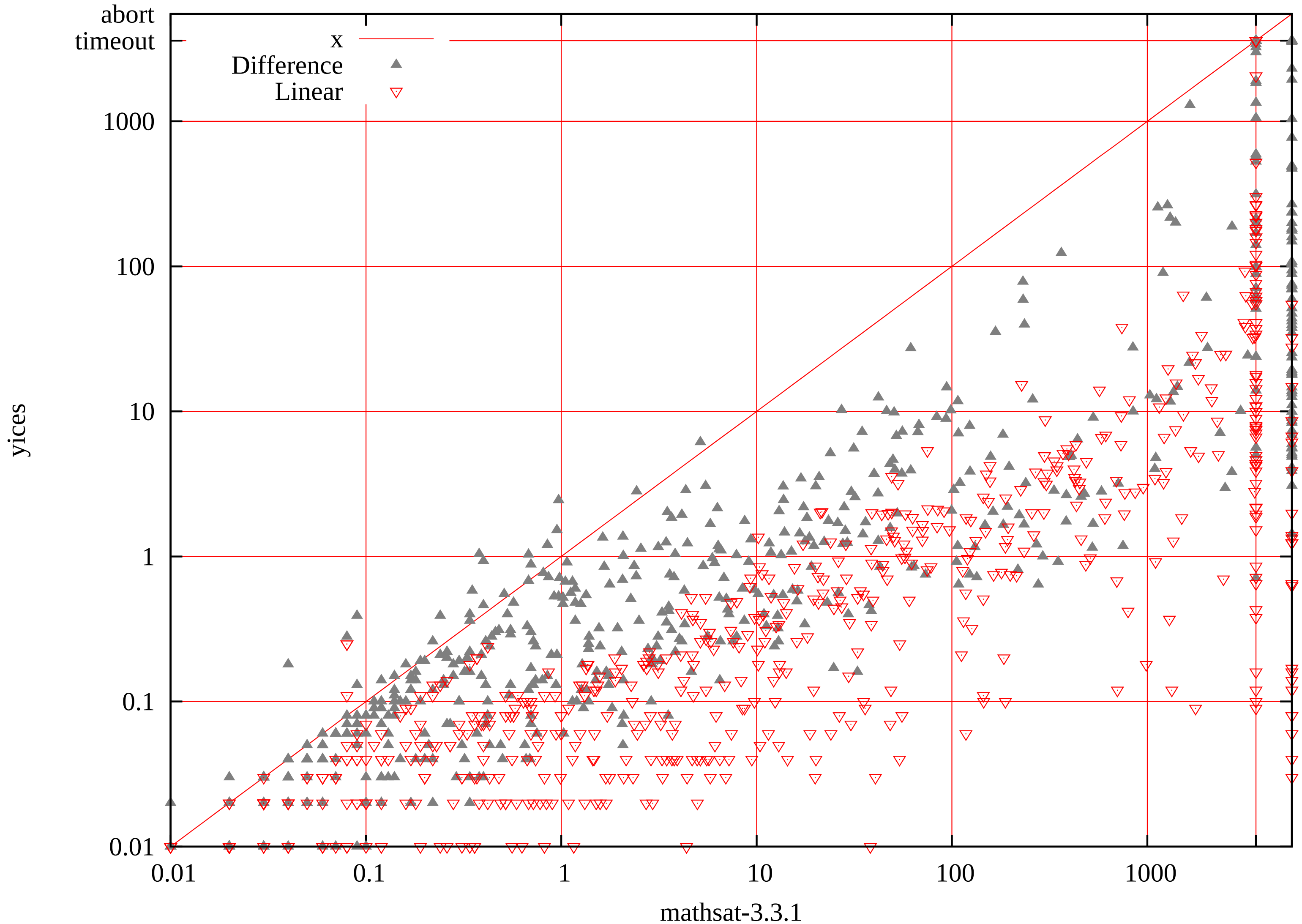
BarcelogicTools 1.0 vs. Yices



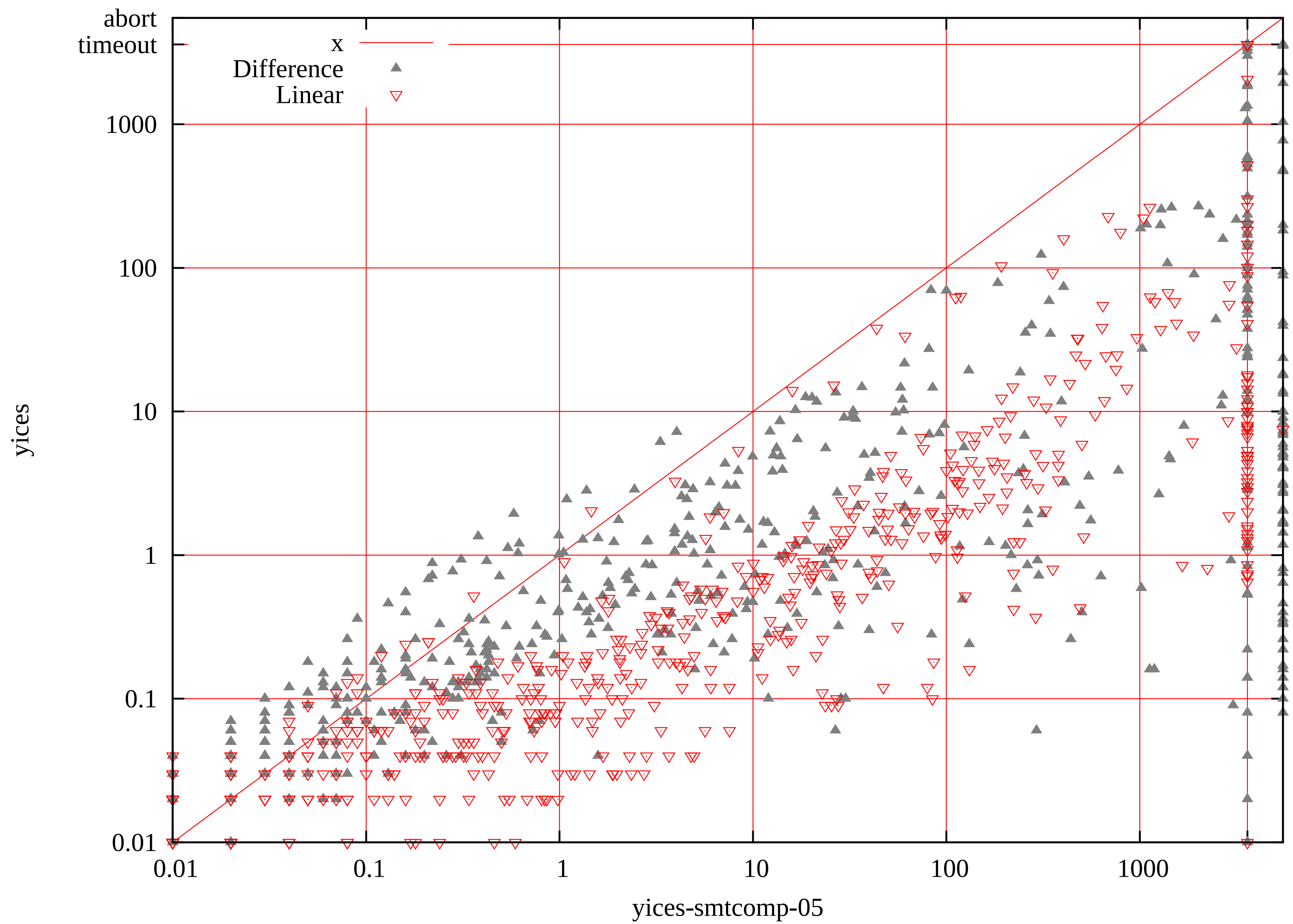
CVC Lite 2.0 vs. Yices



MathSAT 3.3.1 vs. Yices



Old Yices vs. Yices



Conclusion

- We have presented a new algorithm for linear arithmetic:
  - Precise Explanations.
  - Efficient Backtracking.
  - Efficient Theory Propagation.
  - Presimplification Step.
- Outperforms specialized solvers on difference logic.
- The algorithm is used in Yices 1.0.

Conclusion (cont.)

- Yices 1.0 is competing in SMT-COMP'06.
- Yices supports all theories in SMT-COMP and much more.
  - Linear integer & real (& mixed) arithmetic.
  - Extensional arrays
  - Fixed-size bit-vectors
  - Quantifiers
  - Recursive datatypes, tuples, records
  - Lambda expressions
- Yices 1.0 is freely available for end-users.
  - `http://yices.csl.sri.com`